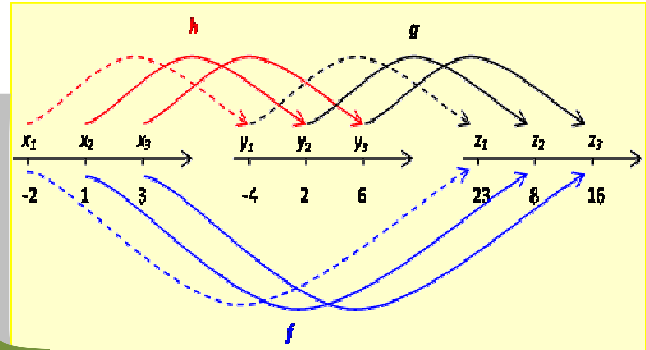
Unit 1



FURTHER ON RELATIONS AND FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *know specific facts about relations.*
- *know additional concepts and facts about functions.*
- *understand methods and principles in composing functions.*

Main Contents

INTRODUCTION

RELATIONSHIPS BETWEEN ELEMENTS OF SETS OCCUR IN MANY CONTEXTS. EXAMPLES OF REL IN SOCIETY INCLUDE ONE PERSON BEING A BROTHER OF ANOTHER PERSON OR ONE PERSON I EMPLOYEE OF ANOTHER.

ON THE OTHER HAND, IN A SET OF NUMBERS, ONE NUMBER BEING A DIVISOR OF ANOTHER, O NUMBER BEING GREATER THAN ANOTHER ARE SOME EXAMPLES OF RELATIONS.

IN GRADES 9 AND 10, YOU LEARNED A GREAT DEAL ABOUT RELATIONS AND FUNCTIONS. IN THIS UNIT YOU WILL STUDY SOME MORE ABOUT THEM. WE HOPE THAT YOUR UNDERSTANDING OF THE WILL BE STRENGTHENED. YOU WILL ALSO STUDY SOME ADDITIONAL TYPES OF FUNCTIONS.

HISTORICAL NOTE

Rene Descartes (1596 - 1650)

Rene Descartes was a philosopher and a mathematician, who assigned coordinates to describe points in a plane. The *xy*-coordinate plane is sometimes called the Cartesian plane in honour of this Frenchman. Descartes' discovery of the Cartesian coordinate system helped the growth of mathematical discoveries for more than 200 years.

John Stuart Mill called Descartes' invention of the Cartesian plane "*The greatest single step ever made in the progress of the exact sciences*".

OPENING PROBLEM

A SET OF GLASSES THAT ARE IN THE SHAPE OF RIGHT CIRCULAR CONES ARE TO BE MADE FOR DISPLAY AS SHOWN IN THE ADJACENT FIGURE. THE GLASSES HAVE THE SAME HEIGHT, $h = 10$ CM. IF THE VOLUME OF A CONE v, AS A FUNCTION OF r, IS GIVEN BY THE FORMULA $v = f(r) = \frac{1}{3} hr^2$

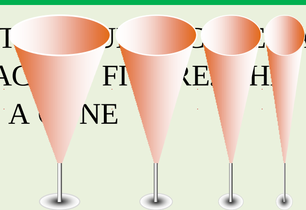


Figure 1.1

- CAN YOU EXPRESS $r = f^{-1}(v)$?
- CAN YOU FILL IN THE FOLLOWING TABLE? MEASUREMENTS ARE IN CM. ROUND r TO TWO DECIMAL PLACES

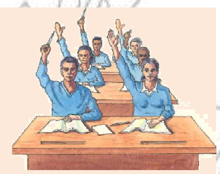
v	40	80	120	160	200	240	280
r							

- CAN YOU DRAW THE GRAPH OF f^{-1}?

1.1 REVISION ON RELATIONS

1.1.1 Inverse of a Relation

ACTIVITY 1.1



1 Let A = {1, 5, 6, 7, 8} and B = {− 1, 2, 4, 9} be two sets and

R = {(5, –1), (6, 4), (7, 9), (8, 2), (1, –1)} be a relation from A to B.

Give the domain and the range of R.

2 Let R = {(x, y) : x < y}. Which of the following ordered pairs belong to R?

A (–5, 6) **B** (, 3.4) **C** (–4, –6.234)

3 Reverse the order of each of the ordered pairs in Questions 1 and 2 above.

Note:

- ✓ A relation is a set of ordered pairs.
- ✓ Given two sets A and B, a relation from A to B is a subset of A × B.
- ✓ A relation on A is any subset of A × A.
- ✓ Let R be a relation from A to B. Then,

  Domain of R = {x ∈ A: (x, y) ∈ R, for some y ∈ B}

  Range of R = {y ∈ B: (x, y) ∈ R, for some x ∈ A}

If R is a relation from A to B, then you may want to know what the inverse of R is. The following definition explains what we mean by the inverse of a relation.

Definition 1.1

Let R be a relation from A to B. The inverse of R, denoted by R^{-1}, is a relation from B to A, given by

$$R^{-1} = \{(b, a): (a, b) \in R\}.$$

Example 1 Let A = {0, –1, 2} and B = {5, 6}.

Give the inverse of R = {(0, 5), (0, 6), (–1, 6)}.

Solution $(a, b) \in R$ means $(b, a) \in R^{-1}$. Thus, $R^{-1} = \{(5, 0), (6, 0), (6, –1)\}$

Example 2 LET A BE THE SET OF ALL TOWNS IN ETHIOPIA, AND B BE THE SET OF ALL REGIONS IN ETHIOPIA. IF R = {(a, b): TOWN a IS FOUND IN REGION b}, THEN FIND R^{-1}.

Solution NOTICE THAT, THE 1ST ELEMENT OF ANY ORDERED PAIR IN R IS A TOWN, WHILE THE 2ND ELEMENT IS A REGION.

THUS, IN R^{-1}, THE 1ST ELEMENT OF THE ORDERED PAIR SHOULD BE A REGION WHILE THE 2ND ELEMENT SHOULD BE A TOWN.

SO, R^{-1} = {(b, a): REGION b CONTAINS TOWN a}

= {(a, b): REGION a CONTAINS TOWN b}

Example 3 LET $R = \{(x, y) : y = x + 3\}$. FIND R^{-1}.

Solution IN R, THE 2ND COORDINATE IS 3 PLUS THE 1ST COORDINATE. THUS,

$$R^{-1} = \{(y, x) : (x, y) \in R\} = \{(x, y) : (y, x) \in R\}.$$

$= \{(x, y) : x = y + 3\}$. NOTICE THAT THE 1ST COORDINATE IS 3 PLUS THE 2ND COORDINATE.

$= \{(x, y) : y = x - 3\}$. SOLVE FOR y.

Example 4 LET R = {(x, y): $y \le x + 3$ AND $y > -2x + 6$}. GIVE R^{-1}.

Solution R^{-1} = {(y, x): $y \le x + 3$ AND $y > -2x + 6$}

= {(x, y): $x \le y + 3$ AND $x > -2y + 6$}

$$= \left\{(x, y) : y \ge x - 3 \text{ AND } y > -\frac{1}{2}x + 3\right\}$$

Group work 1.1

1 IF A = {1, 2, 3, 4, 5} AND B = {u, v, w, x}, THEN WHICH OF THE FOLLOWING ARE RELATIONS FROM A TO B?

- **A** R_1 = { (1, v), (2, w), (5, x)}
- **B** R_2 = { (1, v), (3, 3), (4, v), (4, w)}
- **C** R_3 = { (1, y), (1, x), (3, v), (3, x) }
- **D** R_4 = $\varnothing$

2 FOR THE RELATION IN EXAMPLE 1 ABOVE,

- **A** FIND THE DOMAIN AND RANGE OF R.
- **B** FIND THE DOMAIN AND RANGE OF R^{-1}.
- **C** COMPARE THE DOMAIN OF R WITH THE RANGE OF R^{-1} AND THE RANGE OF R WITH THE DOMAIN OF R^{-1}. WHAT DO YOU NOTICE?

3 FOR THE RELATION GIVEN IN EXAMPLE 2 ON THE PREVIOUS PAGE, IF AMBO TOWN IS IN OROMIA REGION AND JIJIGA TOWN IS IN SOMALI REGION, WHICH OF THE FOLLOWING IS IN R^{-1}?

A (JIJIGA, SOMALI) **B** (OROMIA, JIJIGA)

C (OROMIA, AMBO) **D** (SOMALI, JIJIGA)

4 FOR THE RELATION IN EXAMPLE 4 ON THE PREVIOUS PAGE, FIND THE DOMAIN AND RANGE OF R^{-1}.

5 GIVE THE DOMAIN AND RANGE OF THE INVERSE OF EACH OF THE FOLLOWING RELATIONS.

A $R = \left\{(1,5), (3, -6), (4, 3.5), \left(1, \frac{6}{5}\right)\right\}$

B $R = \{(x, y) : y = 3x - 7\}$

C $R = \{(x, y) : y < -3x$ AND $y \geq x - 4\}$

Exercise 1.1

1 IF $R = \{(x, y): y \geq x + 1\}$, WHICH OF THE FOLLOWING IS TRUE?

A $(0, 0) \in R$ **B** $0 \in$ DOMAIN OF R

C $(0, 1) \in R$ **D** $(-5, 6) \in R$

E $(-5, -5) \in R$ **F** $0 \in$ RANGE OF R.

2 LET $R = \{(x, y) : y \geq x^2 - 1$ AND $y \leq 3\}$

A SKETCH THE GRAPH OF R.

B GIVE THE DOMAIN AND THE RANGE OF R.

3 GIVE THE RELATION REPRESENTED BY THE SHADED REGION IN FIGURE 1.2

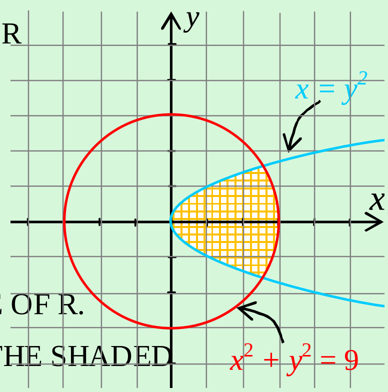


Figure 1.2

4 GIVE THE INVERSE OF EACH OF THE FOLLOWING RELATIONS

A $R = \{(x, y): x$ IS A BROTHER OF $y\}$

B $R = \{(x, y) : x^2 + 1 = y^2\}$

C $R = \{(x, y) : y \geq x + 3$ AND $y < -3x - 1\}$

5 GIVE THE DOMAIN AND RANGE OF THE INVERSE OF EACH OF THE FOLLOWING RELATIONS.

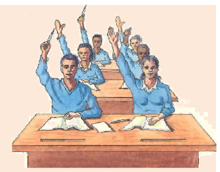
A $R = \{(x, y) : y \geq x^2 + 1\}$

B $R = \{(x, y) : y \leq -x^2$ AND $y \geq -1\}$

C $R = \{(x, y) : -3 \leq x \leq 3, y \in \mathbb{R}\}$

1.1.2 Graphs of Inverse Relations

ACTIVITY 1.2



DO THE FOLLOWING IN PAIRS.

LET R = {(1, –2), (3, 9), (4, 6), (5, –7), (5, 2.5)}

A LIST THE ELEMENTS OF R^{-1}.

B COMPARE THE DOMAIN OF R^{-1} AND THE RANGE OF R. WHAT DO YOU NOTICE?

C COMPARE THE RANGE OF R^{-1} AND THE DOMAIN OF R. WHAT DO YOU NOTICE?

D DO THE SAME FOR R = $\{(x, y): -3 \le x \le 3, y \in \mathbb{R}\}$.

E HOW CAN YOU GENERALIZE YOUR FINDINGS?

FROM WHAT YOU DID SO FAR, YOU SHOULD HAVE CONCLUDED THAT

Domain of R^{-1} = Range of R

Range of R^{-1} = Domain of R

Note:

- ✓ ON THE CARTESIAN COORDINATE PLANE, IN KEEPING WITH COMMON USAGE, ARROWS ARE USED ON THE AXES TO SHOW POSITIVE DIRECTION.
- ✓ IF THE BOUNDARY CURVE IN THE GRAPH IS NOT A PART OF THE RELATION, IT IS SHOWN USING A BROKEN LINE.

NOW, LET US COMPARE GRAPHS OF R AND R^{-1} AND SEE THEIR RELATIONSHIP.

Example 5 LET R = $\{(x, y): y \ge x^2\}$. DRAW THE GRAPH OF R AND R^{-1} USING THE SAME COORDINATE AXES.

Solution $R^{-1} = \{(x, y): x \ge y^2\}$.

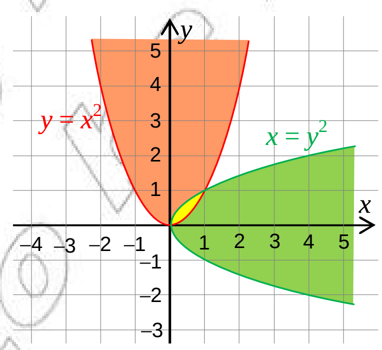


Figure 1.3 *The graph of R and R^{-1}.*

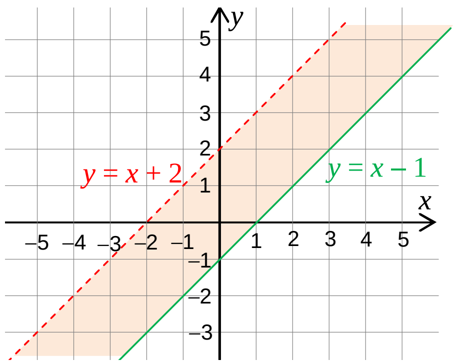
NOTICE THAT $y = x^2$ AND $x = y^2$ MEET AT (0, 0) AND (1, 1). THE EQUATION OF THE LINE THROUGH THE TWO POINTS IS $y = x$.

Example 6 FOR THE FOLLOWING RELATION, SKETCH THE GRAPHS OF R AND R^{-1} USING DIFFERENT COORDINATE AXES.

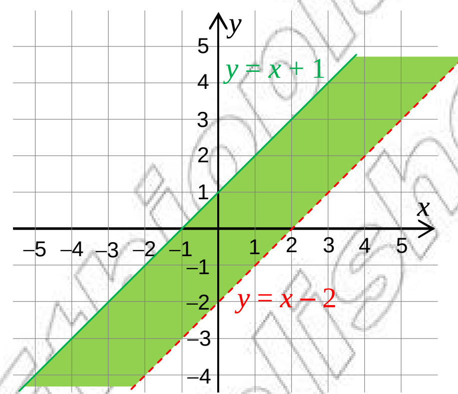
$$R = \{(x, y): y < x + 2 \text{ AND } y \geq x - 1\}.$$

Solution $R^{-1} = \{(x, y): x < y + 2 \text{ AND } x \geq y - 1\}$

$= \{(x, y): y > x - 2 \text{ AND } y \leq x + 1\}$



A *The graph of R*



B *The graph of R^{-1}*

Figure 1.4

Group Work 1.2

1. LET R = {(3, –1), (4, 2), (6, 3), (–5, 1)}
   - **A** LIST THE ELEMENTS OF R^{-1}.
   - **B** ON A PIECE OF SQUARED PAPER, SKETCH THE LINE $y = x$.
   - **C** SKETCH R AND R^{-1} ON THE PAPER, USING DIFFERENT COLOURS (OR MARKING POINTS OF R BY * AND POINTS OF R^{-1} BY Δ).
   - **D** FOLD THE PAPER ALONG THE LINE $y = x$.
   - **E** WHAT DO YOU NOTICE?
2. LET R = $\{(x, y): y = x^3\}$. GIVE R^{-1}. REPEAT THE ABOVE INVESTIGATION B–E FOR R.
3. SKETCH THE GRAPH OF R = $\{(x, y): y < x + 2 \text{ AND } y \geq x - 1\}$ ON SQUARED PAPER; THEN TURN THE PAPER OVER, ROTATE IT 90° CLOCKWISE, AND FINALLY HOLD IT UP TO THE LIGHT. WHAT DO YOU SEE THROUGH THE PAPER? COMPARE IT WITH THE GRAPH OF R^{-1} IN EXAMPLE 6 ABOVE. WHY DOES THIS PROCEDURE WORK?

FROM THE ABOVE GROUP WORK, YOU SHOULD CONCLUDE THAT R AND R^{-1} ARE MIRROR IMAGES OF EACH OTHER ON THE LINE $y = x$. THIS MEANS, IF YOU REFLECT THE GRAPH OF R IN THE LINE $y = x$, YOU GET THE GRAPH OF R^{-1} AND VICE VERSA.

Exercise 1.2

1 **A** LET R = {(*x*, *y*): $x + 1 = y^2$}. DRAW THE GRAPH OF R^{-1} BY REFLECTING THE GRAPH OF R IN THE LINE $y = x$.

B CONSIDER THE FOLLOWING GRAPH OF A RELATION R.

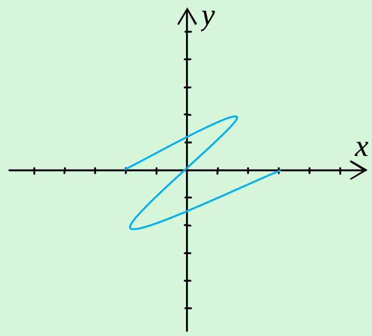
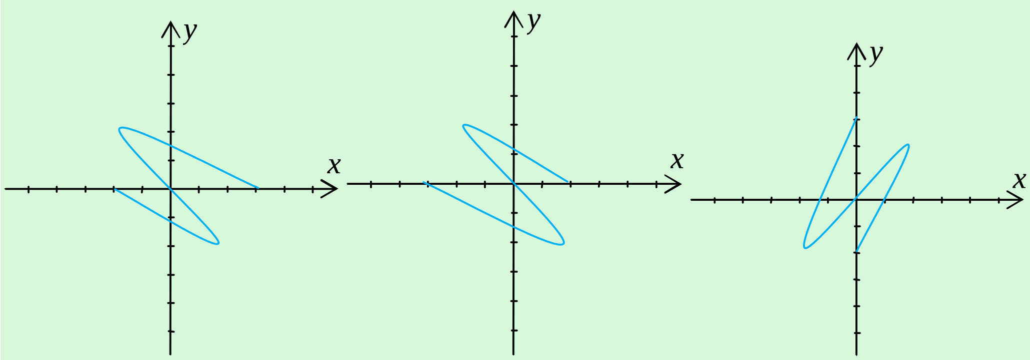


Figure 1.5

WHICH OF THE FOLLOWING IS THE GRAPH OF THE INVERSE OF R?



I II III

Figure 1.6

2 FOR EACH OF THE FOLLOWING RELATIONS, DRAW THE GRAPH OF R AND ITS INVERSE USING THE SAME COORDINATE AXES.

A R = { $(x, y) : x + y \leq 1$}

B R = { $(x, y) : y \geq x + 1$ AND $y < -3x$}

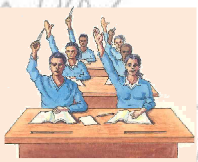
C R = { $(x, y) : x^2 + y^2 = 16$}

D R = { $(x, y) : x^2 + y^2 > 16$}

1.2 SOME ADDITIONAL TYPES OF FUNCTIONS

1.2.1 Revision on Functions

ACTIVITY 1.3



1 WHICH OF THE FOLLOWING ARE FUNCTIONS?

A $R = \{(1,0),(2,0),(3,1),(1,6)\}$ **B** $S = \{(1,0),(2,0),(3,1),(6,1)\}$

C $T = \{(x, y): y = 2x - 1\}$ **D** $W = \{(x, y): y \geq 2x + 9\}$

E $K = \{(1, 3),(3, 2),(1, 7),(-1, 4)\}$ **F** $L = \{(0, 0),(0, -2),(0, 2),(0, 4)\}$

2 LET $f(x) = 9x - 2$ AND $g(x) = \sqrt{3x + 7}$. FIND THE DOMAIN OF f AND g AND EVALUATE THE FOLLOWING:

A $f(-2)$ **B** $f\left(\frac{-1}{2}\right)$ **C** $g(3)$

Note:

- ✓ A FUNCTION IS A RELATION IN WHICH NO TWO DISTINCT ORDERED PAIRS HAVE THE SAME FIRST ELEMENT.
- ✓ IF f IS A FUNCTION WITH DOMAIN A AND RANGE A SUBSET OF B, WE WRITE $f: A \to B$ OR $A \xrightarrow{f} B$
- ✓ IF $f: A \to B$ IS GIVEN BY A RULE THAT MAPS x IN A TO y IN B, THEN WE WRITE $y = f(x)$.

Example 1 SUPPOSE $f: A \to B$ IS THE FUNCTION THAT GIVES $5x - 1$ FOR ANY x IN A. WHAT ARE THE POSSIBLE WAYS OF WRITING THIS FUNCTION?

Solution WE CAN WRITE IT AS

$$f: x \to 5x - 1 \text{ OR } f(x) = 5x - 1 \text{ OR } y = 5x - 1 \text{ OR } x \xrightarrow{f} 5x - 1.$$

Note:

- ✓ $f(x)$ IS READ AS "f OF x".
- ✓ $y = f(x)$, IF AND ONLY IF $P(x, y)$ IS A POINT ON THE GRAPH OF f

Vertical line test:

A SET OF POINTS IN THE CARTESIAN PLANE IS THE GRAPH OF A FUNCTION, IF AND ONLY IF NO V
INTERSECTS THE SET MORE THAN ONCE.

Definition 1.2

A FUNCTION $f: A \to B$ IS SAID TO BE

i ODD, IF AND ONLY IF, FOR ANY $x \in A$, $f(-x) = -f(x)$.

ii EVEN, IF AND ONLY IF, FOR ANY $x \in A$, $f(-x) = f(x)$. THE EVENNESS OR ODDNESS OF A FUNCTION IS CALLED ITS **parity**.

Example 2

a $f(x) = x^3$ IS ODD, SINCE $f(-x) = (-x)^3 = -x^3 = -f(x)$.

b $f(x) = x^2$ IS EVEN SINCE $f(-x) = (-x)^2 = x^2 = f(x)$.

c $f(x) = x + 1$ IS NEITHER EVEN NOR ODD SINCE $f(-x) = -x + 1 \neq -(x + 1) = -f(x)$ AND $f(-x) = -x + 1 \neq x + 1 = f(x)$,

Note:

Exponential and Logarithmic Functions

✓ A FUNCTION $f: \mathbb{R} \to (0, \infty)$ GIVEN BY $f(x) = a^x$, $a > 0$, $a \neq 1$ IS CALLED AN **exponential function**.

✓ A FUNCTION $f: (0, \infty) \to \mathbb{R}$ GIVEN BY $f(x) = \log_a x$, $a > 0$, $a \neq 1$ IS CALLED A **logarithmic function**.

✓ IF $a > 0$, $a \neq 1$, THEN, $\log_a a^x = a^{\log_a x} = x$.

Exercise 1.3

1 DRAW THE GRAPH OF EACH OF THE FOLLOWING FUNCTIONS:

a $f(x) = \frac{3x-1}{2}$ **b** $g(x) = \sqrt{x+1}$ **c** $f(x) = 4$

2 A RESEARCHER INVESTIGATING THE EFFECT OF AIR POLLUTION FOUND THAT THE PERCENTAGE $p(x)$ OF DISEASED TREES AND SHRUBS AT A DISTANCE x KM FROM AN INDUSTRIAL CITY IS GIVEN BY $p(x) = 32 - \frac{3x}{50}$, FOR $50 \leq x \leq 500$. SKETCH THE GRAPH OF THE FUNCTION p AND FIND $p(50)$, $p(100)$, $p(200)$, $p(400)$.

3 DETERMINE WHETHER EACH OF THE FOLLOWING FUNCTIONS IS EVEN, ODD OR NEITHER.

a $g(x) = \sqrt{8x^4 + 1}$ **b** $f(x) = 4x^3 - 5x$

c $f(x) = x^4 + 3x^2$ **d** $h(x) = \frac{1}{x}$

4 USE THE VERTICAL LINE TEST TO DETERMINE THE GRAPH(S) THAT REPRESENT FUNCTION(S).

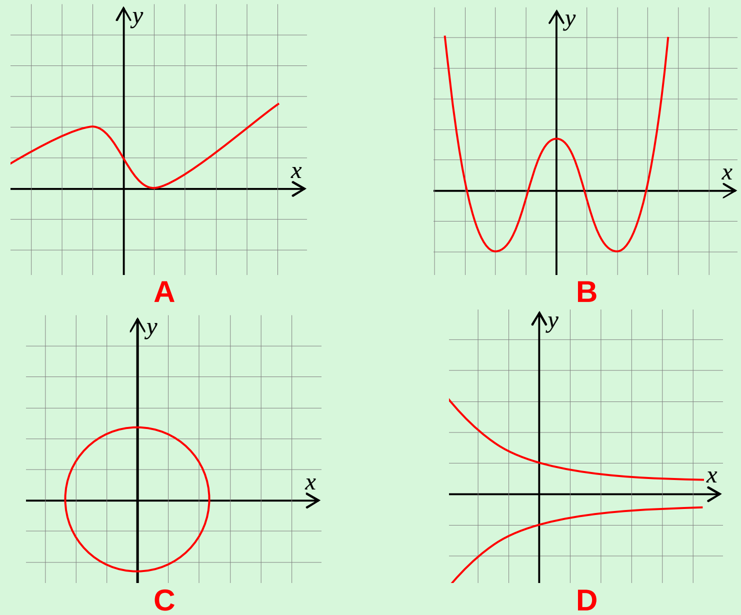
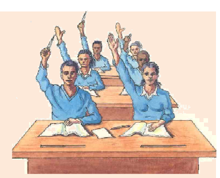


Figure 1.7

1.2.2 Power Functions

ACTIVITY 1.4



GIVEN THE FUNCTIONS

A $f(x) = 2x^7$ **B** $g(x) = x$ **C** $h(x) = 4^x$

D $f(x) = x^{\frac{3}{2}}$ **E** $g(x) = \left(\frac{2}{3}\right)^x$

CLASSIFY EACH AS A POWER FUNCTION OR AN EXPONENTIAL FUNCTION.

Definition 1.3

A POWER FUNCTION IS A FUNCTION WHICH CAN BE WRITTEN IN THE FORM $f(x) = ax^r$, WHERE r IS A RATIONAL NUMBER, $a \in \mathbb{R}$, AND a IS A FIXED NUMBER.

✍Note:

DON'T CONFUSE POWER FUNCTIONS WITH EXPONENTIAL FUNCTIONS.

Exponential function: $y = a^x$ (A FIXED BASE IS RAISED TO A VARIABLE EXPONENT)

Power function: $y = ax^r$ (VARIABLE BASE IS RAISED TO A FIXED EXPONENT)

LET US SEE THE BEHAVIOUR OF A POWER FUNCTION WHEN r IS AN INTEGER.

Group Work 1.3

DO THE FOLLOWING IN GROUPS.

I When r is a positive integer

1 LET $f(x) = 4x^3$

A WHAT IS THE DOMAIN OF f? WHAT IS THE RANGE OF f?

B FILL IN THE FOLLOWING TABLE.

x	–2	–1	0	1	2
$f(x)$					

C SKETCH THE GRAPH OF $f(x)$ USING THE ABOVE TABLE.

D WHAT IS THE PARITY OF f (I.E. IS IT EVEN OR ODD)?

E INVESTIGATE ITS SYMMETRY.

2 GO THROUGH THE STEPS **A** TO **E** FOR THE FUNCTION $f(x) = 4x^2$

II When r is a negative integer

3 LET $f(x) = 2x^{-3}$

A WHAT IS THE DOMAIN OF f? WHAT IS THE RANGE OF f?

B FILL IN THE FOLLOWING TABLE.

x	–2	–1	0	1	2
$f(x)$					

C SKETCH THE GRAPH OF $f(x)$ USING THE ABOVE TABLE.

D WHAT IS THE PARITY OF f (I.E. IS IT EVEN OR ODD)?

E INVESTIGATE ITS SYMMETRY.

4 GO THROUGH THE STEPS **A** TO **E** FOR THE FUNCTION $f(x) = 2x^{-2}$.

WE NOW CONSIDER THE BEHAVIOUR OF A POWER FUNCTION WHEN r IS A RATIONAL NUMBER OF THE FORM $\frac{m}{n}$, WHERE m AND n ARE INTEGERS, WITH $n \neq 0$ (WE WILL ASSUME THAT $\frac{m}{n}$ IS IN ITS LOWEST TERM.)

Example 3 DRAW THE GRAPH OF $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$.

Solution THE FOLLOWING TABLE GIVES SOME VALUES.

x	–8	–1	0	1	8
$f(x)$	–2	–1	0	1	2

USING THE ABOVE VALUES, THE GRAPH CAN BE SKETCHED AS:

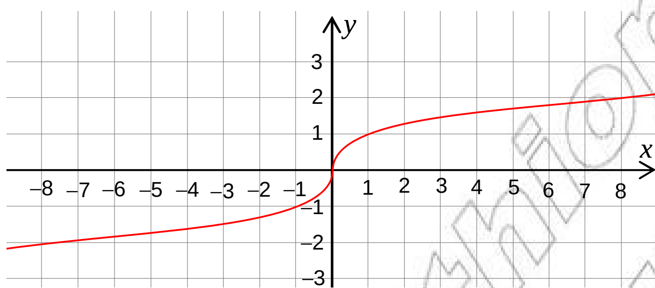


Figure 1.8 *Graph of* $f(x) = \sqrt[3]{x}$.

Note:

1 FOR THE FUNCTION $f(x) = \sqrt[3]{x}$, DOMAIN OF f = RANGE OF f = ℝ.

2 THE POINT (0, 0) WHERE THE GRAPH CHANGES SHAPE FROM CONCAVE UPWARD TO CONCAVE DOWNWARD, IS CALLED AN **inflection point**.

3 ALL FUNCTIONS $f(x) = x^{\frac{1}{n}}$, WHERE n IS AN ODD NATURAL NUMBER, HAVE SIMILAR BEHAVIOURS. THEY ALL PASS THROUGH (–1, –1), (0, 0) AND (1, 1). THEY ARE ALSO INCREASING.

THE FOLLOWING FIGURES GIVE YOU SOME OF THE VARIOUS POSSIBLE GRAPHS OF POWER FU WITH RATIONAL EXPONENTS.

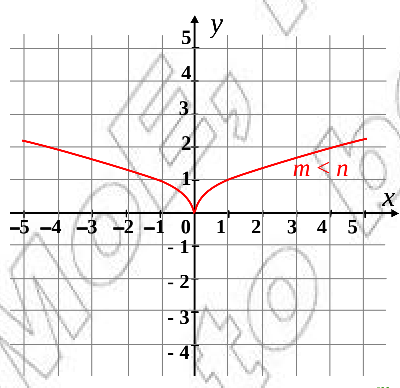


Figure 1.9 *Type of graph of* $y = x^{\frac{m}{n}}$, *m even, n odd*

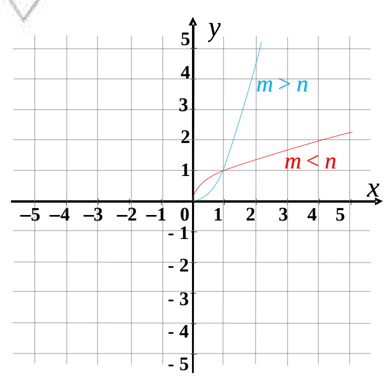


Figure 1.10 *Type of graph of* $y = x^{\frac{m}{n}}$, *m odd, n even*

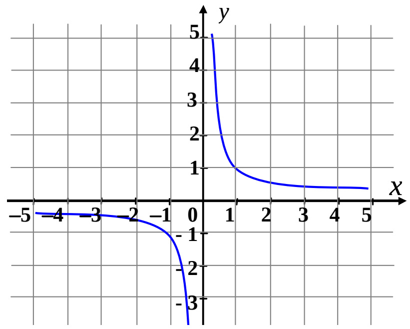


Figure 1.11 *Graph of* $y = x^{\frac{-1}{n}}$, n odd

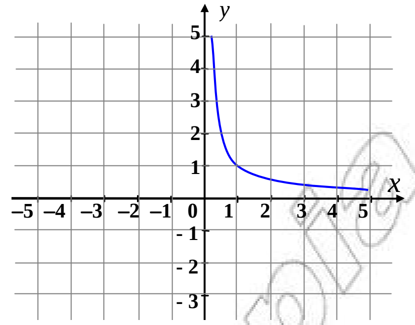


Figure 1.12 *Graph of* $y = x^{\frac{-1}{n}}$, n even

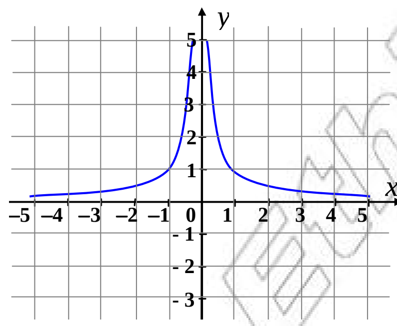
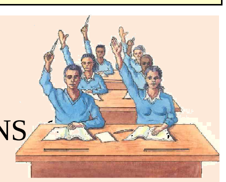


Figure 1.13 *Graph of* $y = x^{\frac{-m}{n}}$, *m even, n odd*

Note:

THE POINT P(0, 0) IN FIGURE 1.9 IS CALLED A **cusp**.

ACTIVITY 1.5



ANSWER EACH OF THE FOLLOWING FOR THE FUNCTIONS IN FIGURES 1.9-1.13 ABOVE.

1 WHAT ARE THEIR DOMAINS AND RANGES?

2 GIVE THEIR PARITIES.

3 STATE WHETHER THEY ARE SYMMETRICAL WITH RESPECT TO THE x-AXIS, THE y-AXIS, THE ORIGIN OR NEITHER.

4 WHERE ARE THEY INCREASING AND WHERE ARE THEY DECREASING?

Exercise 1.4

1 WHICH OF THE FOLLOWING ARE POWER FUNCTIONS AND WHICH ARE NOT (GIVE REASONS).

A $f(x) = 5x^2 + 1$ B $f(x) = 5x^{\frac{-3}{4}}$ C $g(x) = x^{-2}$

D $h(x) = x^x$ E $l(x) = 5^{x+1}$

2 FIND THE DOMAIN OF EACH OF THE FOLLOWING POWER FUNCTIONS:

A $f(x) = x^{\frac{1}{3}}$ **B** $f(x) = x^{\frac{5}{4}}$ **C** $f(x) = 2x^{\frac{-2}{3}}$ **D** $f(x) = x^{\frac{-7}{4}}$

3 SKETCH THE GRAPHS OF f, g AND h USING THE SAME COORDINATE AXES.

$f(x) = x^2$, $g(x) = 2x^2$ AND $h(x) = -2x^2$.

4 IF $f(x) = ax^n$, $a \neq 0$ AND $f(xy) = f(x) f(y)$, WHAT IS THE VALUE OF a?

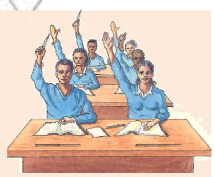
5 CONSIDER $f(x) = ax^{-1}$, $a \neq 0$.

A GIVE THE DOMAIN AND RANGE OF $f(x)$.

B SUPPOSE $a > 0$. THEN $y = f(x)$ CAN BE WRITTEN AS $y = \frac{a}{x}$ OR $xy = a$. HERE x AND y ARE INVERSELY RELATED AND a IS CALLED THE **constant of variation**. DRAW THE GRAPH OF $f(x)$, WHEN $a = 2$, AND DESCRIBE ITS SYMMETRY.

1.2.3 Absolute Value (Modulus) Function

ACTIVITY 1.6



FIND THE ABSOLUTE VALUE OF EACH OF THE FOLLOWING.

A -2 **B** 3 **C** 0 **D** -6.014

Definition 1.4

FOR ANY REAL NUMBER a, THE **absolute value** OR **modulus** OF a, IS DEFINED BY

$$|a| = \begin{cases} a, & \text{IF } a \geq 0 \\ -a, & \text{IF } a < 0 \end{cases}$$

CALCULATOR TIPS

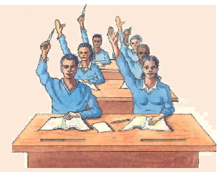
SOME CALCULATORS HAVE KEYS DENOTED BY $|x|$ OR ABS.

YOU CAN USE SUCH A KEY TO FIND THE ABSOLUTE VALUE OF A NUMBER.

IN CASE YOU HAVE A CALCULATOR THAT DOES NOT HAVE SUCH A KEY, TO FIND $|x|$, ENTER x, PRESS THE x^2 KEY, AND THEN PRESS THE $\sqrt{\ }$ KEY.

THIS IS BASED ON THE PROPERTY $|x| = \sqrt{x^2}$

ACTIVITY 1.7



1 COMPARE THE ABSOLUTE VALUES OF

A -3.5 AND 3.5 B 4.213 AND -4.213

C WHAT CAN YOU CONCLUDE ABOUT $|x|$ AND $|-x|$, FOR ANY x IN $\mathbb{R}$?

2 A COMPARE $|xy|$ AND $|x||y|$ FOR THE FOLLOWING.

I $x = 2.4$, $y = 3$ II $x = -6$, $y = 4$

B CONCLUDE WHETHER OR NOT $|xy| = |x||y|$, FOR ALL $x, y \in \mathbb{R}$

Some properties of the absolute value

1 $|x| \geq 0$ FOR ANY $x \in \mathbb{R}$.

2 $|x|$ IS THE DISTANCE BETWEEN THE POINT CORRESPONDING TO x AND THE ORIGIN.

3 $|x| \geq x$ AND $|x| \geq -x$, FOR ANY POINT WITH COORDINATE x IN $\mathbb{R}$.

4 $|x| = |-x|$, FOR ANY $x \in \mathbb{R}$

5 FOR ANY $x, y \in \mathbb{R}$, $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, PROVIDED THAT $y \neq 0$

6 $|xy| = |x||y|$ FOR ANY $x, y \in \mathbb{R}$

7 $|x| = a$, IF AND ONLY IF $x = \pm a$, PROVIDED $a \geq 0$. IN CASE $a < 0$, THEN $|x| = a$ HAS NO SOLUTION.

Definition 1.5

THE **modulus** (**Absolute value**) FUNCTION IS THE FUNCTION f GIVEN BY $f(x) = |x|$

✍Note:

DOMAIN OF $f = \mathbb{R}$. SINCE $f(x) = |x| \geq 0$, FOR EACH $x \in \mathbb{R}$, RANGE OF $f = [0, \infty)$.

Example 4

A COMPLETE THE FOLLOWING TABLE FOR $f(x) = |x|$.

x	-3	-2	-1	0	1	2	3
$f(x)$							

B USING THE ABOVE TABLE, SKETCH THE GRAPH OF $f(x) = |x|$ AND NOTICE ITS FEATURES.

Solution

A

x	–3	–2	–1	0	1	2	3
$f(x)$	3	2	1	0	1	2	3

B FROM THE TABLE, YOU CAN DRAW THE GRAPH AS FOLLOWS:

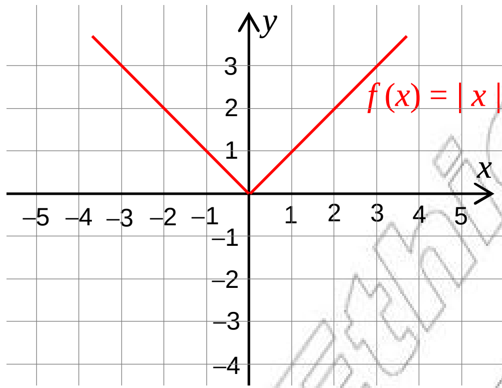


Figure 1.14 *Graph of* $y = |x|$.

AS YOU CAN SEE, THE GRAPH HAS NO HOLE OR BREAK IN IT (I.E. IT IS CONTINUOUS) A MAKES A SHARP CORNER OR A CUSP AT P(0, 0). THE GRAPH IS ALSO SYMMETRICAL V RESPECT TO THE y-AXIS.

Exercise 1.5

1 IF $x = 4$ AND $y = -6$, THEN FIND:

A $|4x - 3|$ **B** $|xy| + 1$ **C** $\dfrac{|x|}{x+1}$

2 GIVE THE SOLUTION SETS FOR EACH OF THE FOLLOWING EQUATIONS.

A $|x| = 4$ **C** $|3x + 1| = 0$

B $|x - 3| = -1$ **D** $|3x + 1| = 5$

3 GIVE THE DOMAIN OF EACH OF THE FOLLOWING FUNCTIONS.

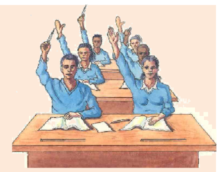
A $f(x) = |x| + 1$ **C** $h(x) = \left|\dfrac{1}{x}\right|$

B $g(x) = |x| - x$ **D** $k(x) = x - \left|\dfrac{x}{2}\right|$

4 SKETCH THE GRAPHS OF $f(x)$ AND $h(x)$ GIVEN IN QUESTION 3 ABOVE.

1.2.4 Signum Function

ACTIVITY 1.8



CONSIDER THE FUNCTION $f(x) = \begin{cases} 3, & \text{IF } x \geq 0 \\ -2, & \text{IF } x < 0 \end{cases}$. FIND

A THE DOMAIN OF f **B** THE RANGE OF f

C SKETCH THE GRAPH OF f

Definition 1.6

THE **signum function**, READ AS SIGNUM x, IS WRITTEN AS SGN x AND IS DEFINED BY

$$y = f(x) = \text{SGN } x = \begin{cases} 1, & \text{FOR } x > 0 \\ 0, & \text{FOR } x = 0 \\ -1, & \text{FOR } x < 0 \end{cases}$$

SINCE, $\dfrac{|x|}{x} = \begin{cases} 1, & x > 0 \\ \nexists \text{ (DOES NOT EXIST)}, & x = 0 \\ -1, & x < 0 \end{cases}$, WE HAVE, SGN $x = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Note:

- ✓ THE SYMBOL $\nexists$ MEANS DOES NOT EXIST OR UNDEFINED.
- ✓ THE SIGNUM FUNCTION IS AN EXAMPLE OF A **piecewise-defined** FUNCTION.
- ✓ IF AN END POINT OF A CURVE IS NOT PART OF THE GRAPH, IT IS SHOWN BY A SMALL OPEN CIRCLE (–∘).
- ✓ IF AN END POINT OF A CURVE IS PART OF THE GRAPH, IT IS SHOWN BY A SMALL FILLED - IN CIRCLE (–•).

Example 5

A COMPLETE THE FOLLOWING TABLE.

x	–4	–3	–2	–1	0	1	2	3	4
sgn x									

B SKETCH THE GRAPH OF $f(x) = $ SGN x USING THE ABOVE TABLE AND FIND ITS DOMAIN AND RANGE .

Solution

a

x	−4	−3	−2	−1	0	1	2	3	4
sgn x	−1	−1	−1	−1	0	1	1	1	1

b

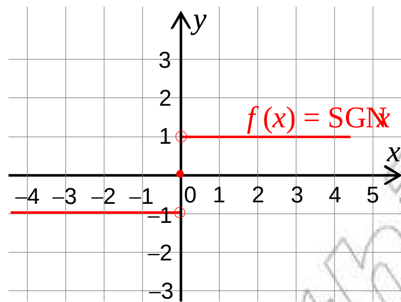


Figure 1.15

AS YOU CAN SEE FROM THE GRAPH, THE DOMAIN OF $f(x) = \text{SGN}\, x$ IS ℝ AND ITS RANGE IS $\{-1, 0, 1\}$.

Exercise 1.6

1 SKETCH THE GRAPH OF $f(x) = x + \text{SGN}\, x$. GIVE THE DOMAIN AND RANGE OF f.

2 DRAW THE GRAPH OF $f(x) = x\ \text{SGN}\, x$. WHAT IS ITS RELATIONSHIP WITH THE GRAPH OF $y = |x|$?

3 SKETCH THE GRAPH OF $f(x) = x^2\ \text{SGN}\, x$. WHAT IS ITS DOMAIN? WHAT IS ITS RANGE?

4 SKETCH THE GRAPH OF $g(x) = x^3\ \text{SGN}\, x$. GIVE ITS DOMAIN AND RANGE. DOES IT HAVE SYMMETRY WITH RESPECT TO ANY LINE?

5 **a** IS $f(x) = \text{SGN}\, x$ EVEN OR ODD? **b** IS $f(x) = x^3\ \text{SGN}\, x$ EVEN OR ODD?

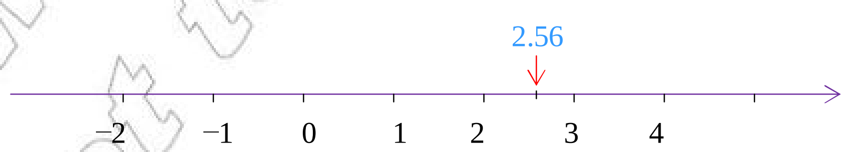
1.2.5 Greatest Integer (Floor) Function

Definition 1.7

THE GREATEST INTEGER FUNCTION, DENOTED BY

$f(x) = \lfloor x \rfloor$, IS DEFINED AS THE **greatest integer** $\leq x$.

Example 6 LET $x = 2.56$. ON THE NUMBER LINE, x IS FOUND BETWEEN 2 AND 3.



WHAT IS THE LARGEST AMONG THE INTEGERS THAT IS LESS THAN OR EQUAL TO 2.56? YOU CAN SEE THAT IT IS 2.

THUS, $\lfloor 2.56 \rfloor = 2$.

Note:

THE GREATEST INTEGER $\le x$ IS ALSO CALLED THE **floor** OF x.

Example 7 FIND $\lfloor x \rfloor$ WHEN

A $x = -4.6$ **B** $x = 3$ **C** $x = 7.2143\ldots$

Solution

A -5 IS THE LARGEST INTEGER ≤ -4.6 I.E., $\lfloor -4.6 \rfloor = -5$.

B 3 IS THE LARGEST INTEGER ≤ 3 I.E. $\lfloor 3 \rfloor = 3$.

C 7.2143 … IS BETWEEN 7 AND 8. THUS, $\lfloor 7.2143 \rfloor = 7$.

ACTIVITY 1.9

1 LET $f(x) = \lfloor x \rfloor$.

A GIVE $f(-3)$, $f(-2.7)$, $f(-2.5)$, $f(-2.1)$, $f(-2.01)$

B WHAT IS $f(x)$, WHEN $-3 \le x < -2$?

C COMPLETE THE FOLLOWING TABLE.

x	$-3 \le x < -2$	$-2 \le x < -1$	$-1 \le x < 0$	$0 \le x < 1$	$1 \le x < 2$	$2 \le x < 3$
$f(x)$	-3	-2				2

2 DRAW THE GRAPH OF $f(x) = \lfloor x \rfloor$.

Note:

THE GREATEST INTEGER $\le x$ IS THE INTEGER THAT IS IMMEDIATELY TO THE LEFT OF x (OR IS x ITSELF, IF x IS AN INTEGER).

AS YOU HAVE SEEN FROM THE EXAMPLES ABOVE, FOR ANY REAL NUMBER x, $\lfloor x \rfloor$ IS ALWAYS AN INTEGER. THUS, DOMAIN = $\mathbb{R}$, RANGE = $\mathbb{Z}$.

WE WRITE THIS AS $f: \mathbb{R} \to \mathbb{Z}$ GIVEN BY $f(x) = \lfloor x \rfloor$.

Exercise 1.7

1 WHAT IS THE VALUE OF EACH OF THE FOLLOWING?

A $\lfloor \ \rfloor$ B $\lfloor -21.01 \rfloor$ C $\lfloor 21.01 \rfloor$ D $\lfloor 0 \rfloor$

2 GIVEN $f(x) = \lfloor x \rfloor$,

I VERIFY THAT $k \in \mathbb{Z}$, $x \in \mathbb{R}$, THEN $f(x + k) = f(x) + k$ BY TAKING

A $x = 4.25, k = 6$ B $x = -3.21, k = 7$ C $x = 8, k = -11$

II VERIFY THAT $f(x) + f(y) \le f(x + y) \le x + y$, USING

A $x = 4.25, y = 6.32$ B $x = -2.01, y =$ C $x = 4, y = -6.24$

III VERIFY THAT $f(x) \le x < f(x) + 1$ BY TAKING

A $x = 2.5$ B $x = -3.54$ C $x = 4$

3 LET $a = x - \lfloor x \rfloor$.

A USING QUESTION 2III ABOVE, SHOW THAT $0 \le a < 1$.

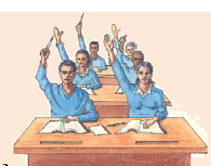
B SHOW THAT $x = \lfloor x \rfloor + a, \ 0 \le a < 1$.

C SHOW THAT $f(x + k) = f(x) + k$, WHEN $k \in \mathbb{Z}$, $x \in \mathbb{R}$ USING 3B.

1.3 CLASSIFICATION OF FUNCTIONS

1.3.1 One-to-One Functions

ACTIVITY 1.10



WHICH OF THE FOLLOWING IS ONE-TO-ONE?

$f = \{(a, 1), (b, 3), (c, 3), (d, 2)\}$; $g = \{(a, 4), (b, 2), (c, 3), (d, 1)\}$

Definition 1.8

A FUNCTION $f: A \to B$ IS SAID TO BE **one-to-one** (**an injection**), IF AND ONLY IF, EACH ELEMENT OF THE RANGE IS PAIRED WITH EXACTLY ONE ELEMENT OF THE DOMAIN, I.E.,

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, FOR ANY x_1 AND $x_2 \in A$.

Note:

THIS IS THE SAME AS SAYING $x_1 \ne x_2 \Rightarrow f(x_1) \ne f(x_2)$.

Example 1 SHOW THAT $f: \mathbb{R} \to \mathbb{R}$ GIVEN BY $f(x) = 2x$ IS ONE-TO-ONE.

Solution LET $x_1, x_2 \in \mathbb{R}$ BE ANY TWO ELEMENTS SUCH THAT $f(x_1) = f(x_2)$.

THEN, $2x_1 = 2x_2 \Rightarrow \frac{1}{2}(2x_1) = \frac{1}{2}(2x_2) \Rightarrow x_1 = x_2$

THUS, f IS ONE-TO-ONE.

Example 2 SHOW THAT $f: \mathbb{R} \to \mathbb{R}$ GIVEN BY $f(x) = x^2$ IS NOT ONE-TO-ONE.

Solution TAKE $x_1 = 2$ AND $x_2 = -2$.

OBVIOUSLY, $x_1 \neq x_2$ I.E $2 \neq -2$

BUT $f(x_1) = f(2) = 2^2 = 4 = (-2)^2 = f(-2) = f(x_2)$

THIS MEANS THERE ARE NUMBERS $x_1, x_2 \in \mathbb{R}$ FOR WHICH $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ DOES NOT HOLD.

THUS, f IS NOT ONE-TO-ONE.

WHEN THE GRAPH OF $f: \mathbb{R} \to \mathbb{R}$ IS GIVEN, I.E. f IS A GRAPHICAL FUNCTION, THERE IS ANOTHER WAY OF CHECKING ITS ONE-TO-ONENESS.

The horizontal line test:

A FUNCTION $f: A \to B$ IS ONE-TO-ONE, IF AND ONLY IF ANY HORIZONTAL LINE CROSSES ITS GRAPH AT MOST ONCE.

Example 3 USING THE HORIZONTAL LINE TEST, SHOW THAT $f: \mathbb{R} \to \mathbb{R}$ GIVEN BY $f(x) = 2x$ IS ONE-TO-ONE.

Solution FROM FIGURE 1.16, IT IS CLEAR THAT ANY HORIZONTAL LINE CROSSES $y = 2x$ AT MOST ONCE. HENCE, $f(x) = 2x$ IS A ONE-TO- ONE FUNCTION.

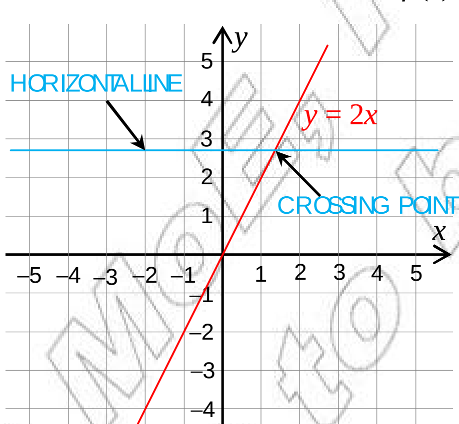


Figure 1.16 *Graph of* $f(x) = 2x$.

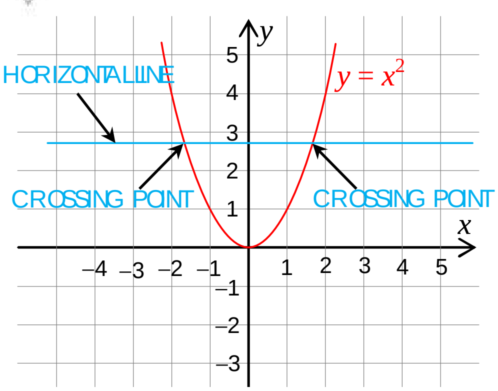


Figure 1.17 *Graph of* $f(x) = x^2$.

Example 4 USING THE HORIZONTAL LINE TEST, SHOW THAT $f: \mathbb{R} \to \mathbb{R}$ GIVEN BY $f(x) = x^2$ IS NOT ONE-TO-ONE.

Solution A HORIZONTAL LINE CROSSES THE GRAPH OF $y = x^2$ AT TWO POINTS (FIGURE 1.17). THUS, f IS NOT ONE- TO- ONE.

Example 5 WHICH OF THE FOLLOWING ARE ONE-TO-ONE FUNCTIONS?

- **A** F = {(x, y): y IS THE FATHER OF x}
- **B** H = {(x, y): $y = |x - 2|$}
- **C** G = {(x, y): x IS A DOG AND y IS ITS NOSE}

Solution ONLY G IS ONE-TO-ONE.

Exercise 1.8

1 WHICH OF THE FOLLOWING FUNCTIONS ARE ONE-TO-ONE?

- **A** $f = \{(1, 5), (2, 6), (3, 7), (4, 8)\}$
- **B** $f = \{(-2, 2), (-1, 3), (0, 1), (4, 1), (5, 6)\}$
- **C** $f = \{(x, y): y$ IS A BROTHER OF $x\}$
- **D** $g = \{(x, y): x$ IS A CHILD AND y IS HIS/H$\}$
- **E** $h: \mathbb{R} \to \mathbb{R}, h(x) = 3x - 2$.
- **F** $h: (0, \infty) \to \mathbb{R}, h(x) = \log_5 x$.
- **G** $f: \mathbb{R} \to \mathbb{R}$, given by $f(x) = |x - 1|$.

2 LET a, b, c, d BE CONSTANTS WITH $ad - bc \neq 0$, AND $f(x) = \dfrac{ax + b}{cx + d}$. CHECK WHETHER OR NOT f IS ONE-TO-ONE.

1.3.2 Onto Functions

Definition 1.9

A FUNCTION $f: A \to B$ IS **onto** (**a surjection**), IF AND ONLY IF, RANGE OF $f = B$.

Example 6 LET f BE DEFINED BY THE VENN DIAGRAM IN FIGURE 1.18 BELOW. RANGE OF $f = B$. THEREFORE, f IS ONTO.

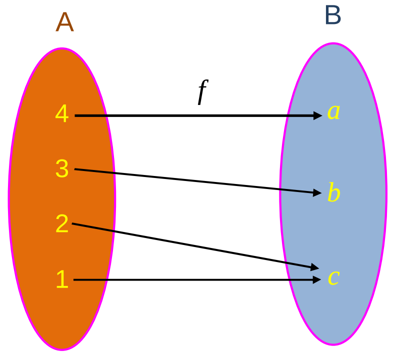


Figure 1.18

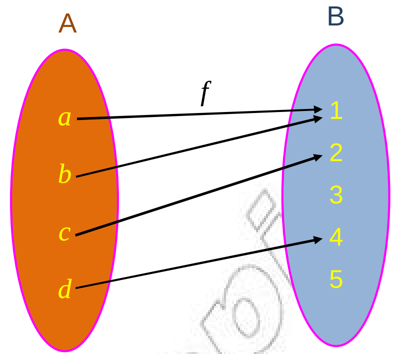


Figure 1.19

Example 7 IN FIGURE 1.19 ABOVE,

RANGE OF $f = \{1, 2, 4\} \Rightarrow$ RANGE OF $f \neq B$

THUS, f IS NOT ONTO.

> **✍Note:**
>
> LET $f: A \to B$ BE A FUNCTION.
>
> RANGE OF f = B MEANS FOR ANY $y \in B$, THERE EXISTS $x \in A$, SUCH THAT $y = f(x)$.
>
> SO, TO SHOW f IS ONTO, IF POSSIBLE, SHOW THAT FOR ANY y, THERE EXISTS x SUCH THAT $f(x) = y$.
>
> TO SHOW f IS NOT ONTO, FIND $y \in B$ THAT IS NOT AN IMAGE OF ANY OF THE ELEMENTS OF A.

Example 8

A LET $f: \mathbb{R} \to \mathbb{R}$ BE $f(x) = x^2$

TAKE $y = -4$. SINCE FOR ANY $x \in \mathbb{R}$, $x^2 \geq 0$, $x^2 \neq -4$. THUS f IS NOT ONTO.

B LET $f: \mathbb{R} \to [0, \infty)$ BE GIVEN BY $f(x) = x^2$.

TAKE $y \in [0, \infty)$. SINCE $y \geq 0$, FOR ALL $x \in \mathbb{R}$, $x^2 \in [0, \infty)$. THUS, $x^2 = y$ HAS A SOLUTION. INDEED, IF $x = \sqrt{y}$, THEN $f(x) = f\left(\sqrt{y}\right) = \left(\sqrt{y}\right)^2 = y$

THUS, f IS ONTO.

> **Definition 1.10**
>
> A FUNCTION f: A $\to$ B IS A **one-to-one correspondence** (**a bijection**), IF AND ONLY IF f IS ONE-TO-ONE AND ONTO.

Example 9 LET $f: \mathbb{R} \to \mathbb{R}$ BE GIVEN BY $f(x) = 5x - 7$. SHOW THAT f IS A ONE-TO-ONE CORRESPONDENCE.

Solution LET $x_1, x_2 \in \mathbb{R}$, SUCH THAT $f(x_1) = f(x_2)$

$$\Rightarrow 5x_1 - 7 = 5x_2 - 7 \Rightarrow 5x_1 - 7 + 7 = 5x_2 - 7 + 7$$

$$\Rightarrow 5x_1 = 5x_2 \Rightarrow x_1 = x_2$$

SO, f IS ONE-TO-ONE.

LET $y \in \mathbb{R}$. IS THERE $x \in \mathbb{R}$ SUCH THAT $y = f(x)$?

IF THERE IS, IT CAN BE FOUND BY SOLVING $y = f(x) = 5x - 7$

$$\Rightarrow y + 7 = 5x \Rightarrow x = \frac{y+7}{5}.$$

SO FOR ANY $y \in \mathbb{R}$, TAKE $x = \frac{y+7}{5} \in \mathbb{R}$.

THEN $f(x) = f\left(\frac{y+7}{5}\right) = 5\left(\frac{y+7}{5}\right) - 7 = y$

SO f IS ONTO.

THEREFORE, f IS A ONE-TO-ONE CORRESPONDENCE.

Example 10 CHECK IF THE FOLLOWING FUNCTION IS A ONE-TO-ONE CORRESPONDENCE.

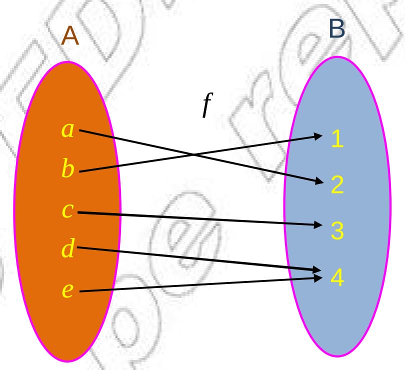


Figure 1.20

Solution f IS ONTO, BECAUSE RANGE OF $f = \{1, 2, 3, 4\} = B$.

BUT f IS NOT ONE-TO-ONE, BECAUSE $f(d) = f(e) = 4$, WHILE $d \neq e$.

SO, f IS NOT A ONE-TO-ONE CORRESPONDENCE.

Example 11 LET $f : \mathbb{R} \to \mathbb{R}$ BE GIVEN BY $f(x) = 3^x$. CHECK WHETHER OR NOT f IS A ONE-TO-ONE CORRESPONDENCE.

Solution FOR ANY $x_1, x_2 \in \mathbb{R}$,

$$f(x_1) = f(x_2) \Rightarrow 3^{x_1} = 3^{x_2} \Rightarrow \frac{3^{x_1}}{3^{x_2}} = 1 \Rightarrow 3^{x_1 - x_2} = 1 = 3^0$$

$$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

THUS, f IS ONE-TO-ONE. BUT f IS NOT ONTO, BECAUSE NEGATIVE NUMBERS CANNOT BE IMAGES. FOR INSTANCE, TAKE $y = -4$.

SINCE $3^x > 0$, FOR EVERY $x \in \mathbb{R}$, IT IS NOT POSSIBLE TO HAVE $x \in \mathbb{R}$ FOR WHICH $3^x = -4$.

THUS, f IS NOT ONTO.

THEREFORE, f IS NOT A ONE-TO-ONE CORRESPONDENCE.

Exercise 1.9

1 WHICH OF THE FOLLOWING FUNCTIONS ARE ONTO?

A $f: \mathbb{R} \to \mathbb{R},\ f(x) = 3x + 5$

B $g: [0, \infty) \to \mathbb{R},\ g(x) = 3 - \sqrt{x}$

C

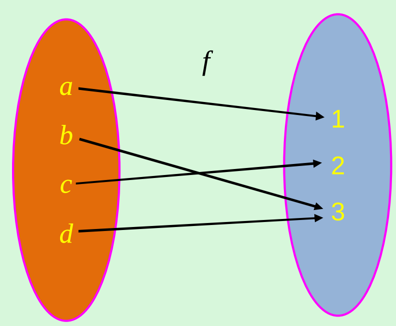


Figure 1.21

D $f: \mathbb{R} \to \mathbb{R},\ f(x) = x^3$

E $h: \mathbb{R} \to \mathbb{R},\ h(x) = |x - 1|$

2 FOR EACH OF THE FOLLOWING FUNCTIONS, GIVE THE SET B FOR WHICH $f: \mathbb{R} \to B$ IS ONTO.

A $f(x) = x^2 + 2$

B $f(x) = |x| + 5$

C $f(x) = 3|x|$

D $f(x) = 1 - 3|x|$

3 SHOW WHETHER EACH OF THE FOLLOWING FUNCTIONS IS A ONE-TO-ONE CORRESPONDENCE OR NOT.

A $f: \mathbb{R} \to \mathbb{R},\ f(x) = \dfrac{3x+1}{5}$

B $g: [0, \infty) \to [0, \infty),\ g(x) = \sqrt{x}$

C $h: \mathbb{R} \to (0, \infty),\ h(x) = 5^x$

D $f: [1, \infty) \to [0, \infty),\ f(x) = (x-1)^2 + 1$

4 FIND A ONE-TO-ONE CORRESPONDENCE BETWEEN EACH PAIR OF THE FOLLOWING SETS.

A $A = \{a, b, c\}$ AND $B = \{1, 2, 3\}$

B $A = \{-1, -2, -3, \ldots, -50\}$, $B = \{1, 2, 3, \ldots, 50\}$.

C $A = \mathbb{N}$ AND $B = \{5, 8, 11, \ldots\}$

1.4 COMPOSITION OF FUNCTIONS

Combination of functions

✍Note:

RECALL THE FOLLOWING.

✓ LET f AND g BE TWO FUNCTIONS. THEN,

$(f+g)(x) = f(x) + g(x)$; $(f-g)(x) = f(x) - g(x)$;

$(fg)(x) = f(x)\,g(x)$; $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$; WHERE $g(x) \neq 0$.

✓ DOMAIN OF $(f+g)$ = DOMAIN OF $(f-g)$

= DOMAIN OF (fg) = DOMAIN OF $f \cap$ DOMAIN OF g

✓ DOMAIN OF $\left(\dfrac{f}{g}\right)$ = (DOMAIN OF $f \cap$ DOMAIN OF g) $\setminus \{x: g(x) = 0\}$

Definition 1.11

LET $f: A \to B$ AND $g: B \to C$ BE FUNCTIONS. THEN, THE COMPOSITION OF g BY f, DENOTED BY gof, IS GIVEN AS $(gof)(x) = g(f(x))$.

Example 1 GIVEN THE VENN DIAGRAM IN FIGURE 1.22, FIND

A $(gof)(a)$ **B** $(gof)(d)$

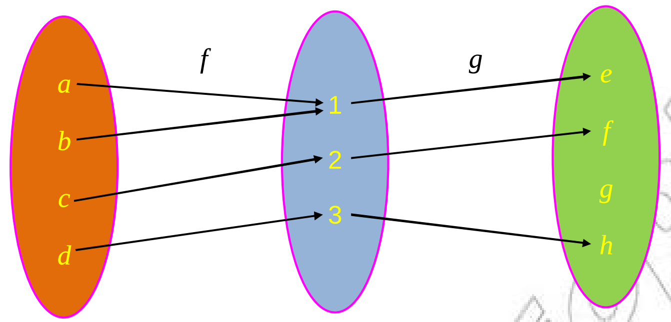


Figure 1.22

Solution $(gof)(a) = g(f(a)) = g(1) = e$ AND $(gof)(d) = g(f(d)) = g(3) = h$

Example 2 GIVEN THE VENN DIAGRAM IN FIGURE 1.23, FIND

A $(gof)(b)$ **B** $(gof)(c)$

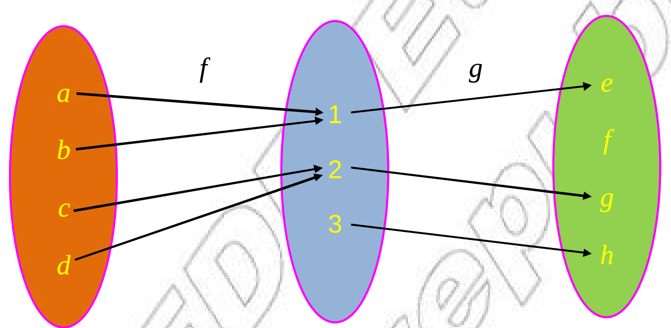


Figure 1.23

Solution $(gof)(b) = g(f(b)) = g(1) = e$ AND $(gof)(c) = g(f(c)) = g(2) = g$

Example 3 LET $f(x) = 2x + 1$, $g(x) = x^3$. GIVE $(fog)(x)$ AND $(gof)(x)$.

Solution $(fog)(x) = f(x^3) = 2x^3 + 1$, WHILE $(gof)(x) = g(2x + 1) = (2x + 1)^3$

Example 4 GIVE $(fog)(x)$, $(gof)(x)$, $(fof)(x)$, $(gog)(x)$, IF THEY EXIST, FOR

$f(x) = \log_5 x$, $g(x) = x^2 + 2$.

Solution RANGE OF $f = \mathbb{R}$ DOMAIN OF $g = \mathbb{R}$.

$\Rightarrow (gof)(x)$ EXISTS AND $(gof)(x) = g(\log_5 x) = (\log_5 x)^2$.

RANGE OF $g = [2, \infty)$. DOMAIN OF $f = (0, \infty)$ AND HENCE

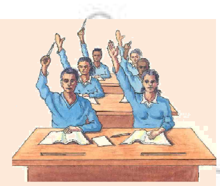
DOMAIN OF $f \cap$ RANGE OF $g \neq \varnothing$.

$\Rightarrow fog$ EXISTS AND $(fog)(x) = \log_5(x^2 + 2)$

$(fof)(x) = \log_5 f(x) = \log_5(\log_5 x)$ CAN BE DEFINED ONLY IF $\log_5 x > 0$ I.E. IF AND ONLY IF $x > 1$.

$(gog)(x) = g(x^2 + 2) = (x^2 + 2)^2 + 2$. HERE x CAN BE ANY REAL NUMBER.

ACTIVITY 1.11



1 LET $f(x) = x + 1$ AND $g(x) = \sqrt{x}$.

A GIVE fog AND gof.

B FIND THE

I DOMAIN OF fog **II** DOMAIN OF gof

III RANGE OF fog **IV** RANGE OF gof

2 LET $f(x) = x^2 - 1$ AND $g(x) = |x|$

A GIVE fog, gof.

B WHAT IS THE DOMAIN OF gof?

C SKETCH THE GRAPH OF gof.

3 LET $f(x) = \log_2 x$ AND $g(x) = |x|$.

A GIVE fog AND gof.

B GIVE THE DOMAIN OF fog AND THE DOMAIN OF gof.

C SKETCH THE GRAPH OF BOTH THE FUNCTIONS fog AND gof.

Example 5 LET $f: \mathbb{R} \to [0, \infty)$ BE GIVEN BY $f(x) = 2^x$ AND $g: [0, \infty) \to [0, \infty)$ BE GIVEN BY $g(x) = \sqrt{x}$. THEN, FIND $(gof)(x)$ AND THE DOMAIN OF $(gof)(x)$.

Solution $(gof)(x) = g(2^x) = \sqrt{2^x} = 2^{\frac{x}{2}}$. DOMAIN OF gof = DOMAIN OF $f = \mathbb{R}$.

Example 6 LET $f(x) = 5x + 4$ AND $gof(x) = 7x - 1$. FIND $g(x)$.

Solution SINCE gof AND f ARE LINEAR, TRY A LINEAR FUNCTION $g(x) = ax + b$.

$g(f(x)) = g(5x + 4) = a(5x+4)+B = 5ax + 4a + b$

NOW, $g(f(x)) = 7x - 1 \Rightarrow 5ax + 4a + b = 7x - 1 \Rightarrow 5a = 7 \Rightarrow a = \frac{7}{5}$ AND

$4a + b = -1 \Rightarrow b = -1 - 4a \Rightarrow b = -1 - \frac{28}{5} = -\frac{33}{5}$

THUS $g(x) = \frac{7}{5}x - \frac{33}{5}$.

Exercise 1.10

1 LET $f(x) = 9x - 2$ AND $g(x) = \sqrt{3x+7}$. EVALUATE THE FOLLOWING.

A $g(3) - g(-2)$ **B** $(g(-1))^2$ **C** $\dfrac{f(x) - f(0)}{x}$

2 LET $f(x) = 9x - 2$; $g(x) = \sqrt{3x+7}$. FIND EACH OF THE FOLLOWING.

A $(f + g)(-2)$ **B** $\dfrac{f}{g}(7)$ **C** DOMAIN OF $(f + g)$

D DOMAIN OF fg **E** DOMAIN OF $\dfrac{f}{g}$

3 FIND $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$ AND $\left(\dfrac{f}{g}\right)(x)$ FOR THE FOLLOWING.

A $f(x) = \dfrac{5x}{2x-1}$; $g(x) = \dfrac{-6x}{2x-1}$ **B** $f(x) = \sqrt{x+1}$; $g(x) = \dfrac{1}{\sqrt{x+1}}$

4 LET $f(x) = 3x - 2$; $g(x) = 5x + 1$. COMPUTE THE INDICATED VALUES.

A $(fog)(3)$ **B** $(fof)(0)$ **C** $(gof)(-5)$

D $(gog)(-7)$ **E** $(fogof)(2)$

5 FIND

I $(fog)(x)$ **II** $(gof)(x)$

III $(fof)(x)$ **IV** $(gog)(x)$, IF THEY EXIST, FOR

A $f(x) = 2x - 1$; $g(x) = 4x + 2$ **B** $f(x) = x^2$; $g(x) = \sqrt{x}$

C $f(x) = 1 - 5x$; $g(x) = |2x + 3|$ **D** $f(x) = 3x$; $g(x) = 2^x$

6 LET $f(x) = 3x$, $g(x) = |x|$ AND $h(x) = \sqrt{x}$. EXPRESS EACH FUNCTION BELOW AS A COMPOSITION OF ANY TWO OF THE ABOVE FUNCTIONS.

A $l(x) = \sqrt{3x}$ **B** $k(x) = 3|x|$ **C** $t(x) = \sqrt{|x|}$

7 EXPRESS EACH FUNCTION f AS A COMPOSITE OF TWO SIMPLER FUNCTIONS h AND g. I.E. $f = hog$.

A $f(x) = \sqrt{3x+1}$ **B** $f(x) = 16x^2 - 3$ **C** $f(x) = 2^{3x^2+1}$

D $f(x) = 5 \times 2^{2x} + 3$ **E** $f(x) = x^4 - 6x^2 + 6$

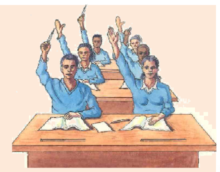
8 LET $f(x) = 4x + 1$ AND $g(x) = 3x + k$, FIND THE VALUE OF k FOR WHICH $(fog)(x) = (gof)(x)$.

9 IF $f(x) = ax + b$, $a \neq 0$, FIND g SUCH THAT $(gof)(x) = x$.

10 GIVEN $f(x) = x^4$ AND $g(x) = 2x + 3$, SHOW THAT $(fog)(x) \neq (gof)(x)$, IN GENERAL.

1.5 INVERSE FUNCTIONS AND THEIR GRAPHS

ACTIVITY 1.12



GIVE THE INVERSES OF EACH OF THE FOLLOWING:

A $f = \{(x, y): y = 3x - 4\}$. IS f^{-1} A FUNCTION?

B $R = \{(x, y): y \geq 3x - 4\}$. IS R^{-1} A FUNCTION?

C $f = \{(x, y): y = x^2\}$. IS f^{-1} A FUNCTION?

D $g = \{(x, y): y = \log_2 x\}$. IS g^{-1} A FUNCTION?

FROM YOUR INVESTIGATION, YOU SHOULD HAVE NOTICED THAT:

Note:

f^{-1} IS A FUNCTION, IF AND ONLY IF f IS ONE-TO-ONE.

Example 1 IS THE INVERSE OF $f(x) = x^3 - x + 1$ A FUNCTION?

Solution DOMAIN OF $f = \mathbb{R}$ AND FOR $1, -1 \in$ DOMAIN OF f

$f(1) = 1 - 1 + 1 = 1 = f(-1)$. THIS IMPLIES f IS NOT ONE-TO-ONE.

THEREFORE f^{-1} IS NOT A FUNCTION.

Notation: IF THE INVERSE OF f IS g, THEN g IS DENOTED BY f^{-1}. IN THIS CASE f IS CALLED **invertible**.

Steps to find the inverse of a function *f*

1. INTERCHANGE x AND y IN THE FORMULA OF f.
2. SOLVE FOR y IN TERMS OF x.
3. WRITE $y = f^{-1}(x)$.

Example 2 FIND THE INVERSE OF EACH OF THE FOLLOWING FUNCTIONS

A $f(x) = 4x - 3$. **B** $f(x) = 1 - 3x$ **C** $f(x) = \dfrac{x}{x-1}, x \neq 1$.

Solution

A $f = \{(x, y): y = 4x - 3\}$ AND

$$f^{-1} = \{(x, y): x = 4y - 3\} = \left\{(x, y): \frac{x+3}{4} = y\right\} \Rightarrow f^{-1}(x) = \frac{x+3}{4}$$

B $f = \{(x, y): y = 1 - 3x\}$

$$\Rightarrow f^{-1} = \{(x, y): x = 1 - 3y\} = \left\{(x, y): y = \frac{1-x}{3}\right\}.$$

THEREFORE, $f^{-1}(x) = \dfrac{1-x}{3}$.

C $f = \{(x, y): y = \dfrac{x}{x-1},\ x \neq 1\}$

$$f^{-1} = \{(x, y): x = \frac{y}{y-1}, x \neq 1\} = \{(x, y): x(y-1) = y, x \neq 1\}$$

$$= \{(x, y): y(x-1) = x, x \neq 1\}$$

$$= \{(x, y): y = \frac{x}{x-1}, x \neq 1\}$$

Definition 1.12

THE FUNCTION $I: A \to A$, GIVEN BY $I(x) = x$ IS CALLED THE **identity function**.

✍Note:

IF $f: A \to A$, AND $I: A \to A$, THEN $(I \circ f)(x) = I(f(x)) = f(x)$, FOR EVERY x.

AGAIN $(f \circ I)(x) = f(I(x)) = f(x)$, FOR EVERY x.

WE CAN DEFINE THE INVERSE OF A FUNCTION USING THE COMPOSITION OF FUNCTIONS AS FO

Definition 1.13

A FUNCTION g IS SAID TO BE AN **inverse** OF A FUNCTION f, IF AND ONLY IF,

$g(f(x)) = I(x)$ AND $f(g(x)) = I(x)$

Example 3 SHOW WHETHER OR NOT EACH OF THE FOLLOWING PAIRS OF FUNCTIONS ARE INVERSES OF EACH OTHER.

A $f: \mathbb{R} \to (0, \infty)$ GIVEN BY $f(x) = 2^x$ AND

$g: (0, \infty) \to \mathbb{R}$ GIVEN BY $g(x) = \log_2 x$.

B $f(x) = \dfrac{x+1}{x+2}, x > -2$ AND $g(x) = \dfrac{1-2x}{x-1},\ x \neq 1$

C $f(x) = \dfrac{x+5}{x+1}; x \neq -1$ AND $g(x) = \dfrac{5-x}{x+1}; x \neq -2$

Solution

a $(f\circ g)(x) = 2^{\log_2 x} = x$ AND $(g\circ f)(x) = \log_2 2^x = x = I(x)$

THUS f AND g ARE INVERSES OF EACH OTHER, I.E. $g = f^{-1}$ OR $f = g^{-1}$

b $f(g(x)) = f\left(\dfrac{1-2x}{x-1}\right) = x = \mathrm{I}(x)$ AND $g(f(x)) = g\left(\dfrac{x+1}{x+2}\right) = x = \mathrm{I}(x)$.

THUS f AND g ARE INVERSES OF EACH OTHER, I.E. $g = f^{-1}$ OR $f = g^{-1}$.

c $f(g(x)) = f\left(\dfrac{5-x}{x+2}\right) = \dfrac{4x+15}{7} \neq \mathrm{I}(x)$ AND

$g(f(x)) = g\left(\dfrac{x+5}{x+1}\right) = \dfrac{4x}{3x+7} \neq \mathrm{I}(x)$

HENCE f AND g ARE NOT INVERSES OF EACH OTHER.

ACTIVITY 1.13

RECALL THAT THE GRAPH OF THE INVERSE OF A RELATION IS ... BY REFLECTING THE GRAPH OF THE RELATION WITH RESPECT T... $y = x$.

FOR EACH OF THE FOLLOWING, SKETCH THE GRAPH OF f AND f^{-1} USING THE SAME COORDINATE AXES.

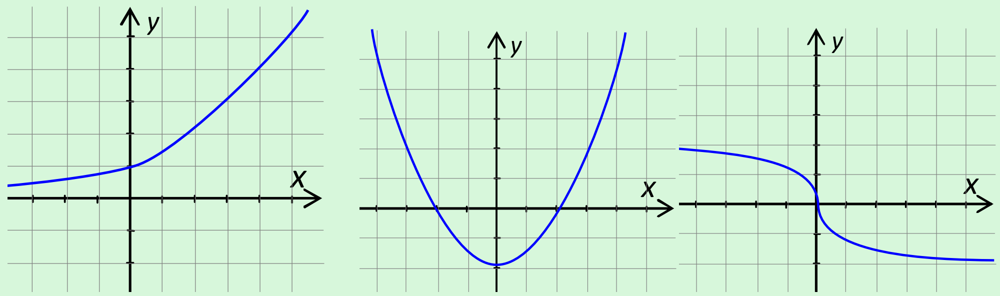
a $f(x) = 2x + 3$ **b** $f(x) = x^3$

FROM ACTIVITY 1.14, YOU MAY HAVE OBSERVED THAT THE GRAPH OF f^{-1} CAN BE OBTAINED BY REFLECTING THE GRAPH OF f WITH RESPECT TO THE LINE $y = x$.

Exercise 1.11

1. DETERMINE THE INVERSE OF EACH OF THE FOLLOWING FUNCTIONS. IS THE INVERSE A FUNCTION?
   - **a** $f(x) = \log_3 2x$ **b** $h(x) = -5x + 13$
   - **c** $g(x) = 1 + \sqrt{x}$ **d** $k(x) = (x-2)^2$
2. GIVE THE DOMAIN OF EACH INVERSE IN QUESTION 1 ABOVE.
3. ARE THE FOLLOWING FUNCTIONS INVERSES OF EACH OTHER (*in their respective domain*)?
   - **a** $f(x) = 3x + 2;\ g(x) = \dfrac{x-2}{3}$ **b** $f(x) = x^3;\ g(x) = \sqrt[3]{x}$
   - **c** $f(x) = \sqrt{x};\ g(x) = x^2$ **d** $f(x) = \sqrt[3]{x+8}$ AND $g(x) = x^3 -$
4. WHICH OF THE FOLLOWING FUNCTIONS ARE INVERTIBLE? IF THEY ARE NOT, CAN YOU RESTRICT THE DOMAIN TO MAKE THEM INVERTIBLE?
   - **a** $f(x) = x^3$ **b** $g(x) = 4 - x^2$
   - **c** $h(x) = -\dfrac{1}{3}x + 5$ **d** $f(x) = \log x^2$

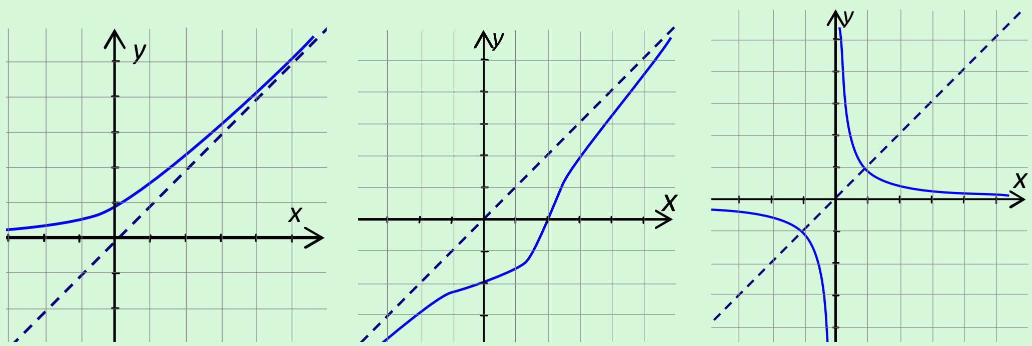
5 WHICH OF THE FOLLOWING FUNCTIONS ARE INVERTIBLE?



A **B** **C**

Figure 1.24

6 SKETCH f^{-1} FOR EACH OF THE FOLLOWING FUNCTIONS.



A **B** **C**

Figure 1.25

Key Terms

combination of functions	modulus (absolute value)
composite function	one-to-one correspondence
cusp	one-to-one function
domain	onto function
function	parity
greatest integer (floor) function	power function
horizontal line test	range
identity function	relation
inflection point	signum (sgn) function
inverse function	vertical line test

Summary

1 A **relation** FROM A TO B IS ANY SUBSET OF A × B.

2 $(f \pm g)(x) = f(x) \pm g(x)$; $(fg)(x) = f(x)g(x)$; $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ PROVIDED THAT $g(x) \neq 0$.

3 $R^{-1} = \{(b, a): (a, b) \in R\}$

4 DOMAIN OF R^{-1} = RANGE OF R AND RANGE OF R^{-1} = DOMAIN OF R.

5 A **function** IS A RELATION IN WHICH NO TWO OF THE ORDERED PAIRS HAVE THE SAME FIRST ELEMENT.

6 $f(x) = ax^r$, $r \in \mathbb{Q}$ IS CALLED A **power function**.

7 $f(x) = ax^{\frac{m}{n}}$, m EVEN AND n ODD HAS A **cusp** AT THE ORIGIN.

8 $|x| = \begin{cases} x, & \text{FOR } x \geq 0 \\ -x, & \text{FOR } x < 0 \end{cases}$

9 $|x| = \sqrt{x^2}$

10 SGN $x = \begin{cases} 1, & \text{FOR } x > 0 \\ 0, & \text{FOR } x = 0 \\ -1, & \text{FOR } x < 0 \end{cases}$

11 THE **floor function** OR THE GREATEST INTEGER FUNCTION $f(x) = \lfloor x \rfloor$ MAPS $\mathbb{R}$ INTO $\mathbb{Z}$.

12 f IS **one-to-one**, IF AND ONLY IF $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, FOR ANY $x_1, x_2 \in$ DOMAIN OF f.

13 A NUMERICAL FUNCTION f IS ONE-TO-ONE, IF AND ONLY IF NO HORIZONTAL LINE CROSSES THE GRAPH OF f MORE THAN ONCE.

14 $f: A \to B$ IS **onto**, IF AND ONLY IF RANGE OF f = B.

15 $f: A \to B$ IS A **one-to-one correspondence**, IF AND ONLY IF f IS ONE-TO-ONE AND ONTO.

16 $(fog)(x) = f(g(x))$

17 DOMAIN OF $(fog) \subseteq$ DOMAIN OF g.

18 f^{-1} IS A FUNCTION, IF f IS ONE-TO-ONE.

19 g AND f ARE INVERSE FUNCTIONS OF EACH OTHER, IF AND ONLY IF $g(f(x)) = x$ AND $f(g(x)) = x$.

20 TO FIND f^{-1}

- ✓ WRITE $y = f(x)$.
- ✓ INTERCHANGE x AND y IN THE ABOVE EQUATION TO OBTAIN $x = f(y)$.
- ✓ SOLVE FOR y AND WRITE $y = f^{-1}(x)$.

Review Exercises on Unit 1

1 FIND THE INVERSE OF EACH RELATION AND DETERMINE WHETHER THE INVERSE IS A FUNCTION.

A R = {(2, –2), (–3, 3), (–4, –4)} **B** R = {(2, 1), (2, 3), (2, 7)}

2 FIND THE INVERSE OF EACH FUNCTION. IS THE INVERSE A FUNCTION?

A $f(x) = 2x + 3$ **B** $f(x) = x^2 - 9$

C $f(x) = (x^2 - 9)^2$ **D** $f(x) = \frac{\sqrt{x}}{3}$

3 FIND THE DOMAIN OF $f(x) = \sqrt{|x| - x}$.

4 **A** GIVE THE INTERSECTION POINTS OF $y = x^5$ AND $y = x^7$.

B ARE THESE POINTS COMMON TO $y = x^n$, WHERE n IS AN ODD NATURAL NUMBER?

C FOR EACH OF THE FOLLOWING FUNCTIONS, WHAT IS THE EFFECT OF 4, IF $y = f(x)$ IS COMPARED WITH $y = x^3$?

I $f(x) = 4x^3$ **II** $f(x) = x^3 + 4$

D FOR $f(x) = x^3$, COMPARE $f(a \cdot b)$ AND $f(a) \cdot f(b)$ FOR ANY $a, b \in \mathbb{R}$. WHAT DO YOU NOTICE?

E IS THE PROPERTY $f(ab) = f(a) \cdot f(b)$ FOR ANY $a, b \in \mathbb{R}$ GENERALLY TRUE FOR ANY $f(x) = x^n, n \in \mathbb{R}$?

5 DRAW THE GRAPH OF $y = |f(x)|$ USING THE GRAPH OF $y = f(x)$, FOR EACH OF THE FOLLOWING:

A $f(x) = x + 1$ **B** $f(x) = \text{LOG } x$ **C** $f(x) = x^3$

6 **A** SHOW THAT $y = \text{SGN } x$ IS ODD.

B IF $h(x) = \frac{1}{2}(\text{SGN } x + 1)$, SHOW THAT $h(-x) + h(x) = 1$

C EXPRESS $h(x)$ IN TERMS OF x, BY TAKING $x > 0$, $x = 0$ AND $x < 0$.

7 FOR $f(x) = \lfloor x \rfloor$, VERIFY THAT $f(x + y) \le f(x) + f(y) + 1$, BY TAKING

A $x = -3.9$; $y = -16.4$ **B** $x = 3.9$; $y = -16.4$

C $x = -3.9$; $y = 16.4$ **D** $x = 3.9$; $y = 16.4$

8 CHECK IF $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$, BY TAKING DIFFERENT VALUES OF x.

9 FIND fog, fof, gof, gog FOR

A $f(x) = 1 + 2x$; $g(x) = |x|$ **B** $f(x) = \text{LOG } x$; $g(x) = 3x + 1$

10 WHAT IS THE DOMAIN OF EACH COMPOSITE FUNCTION IN QUESTION 9?

11 DETERMINE WHETHER OR NOT EACH PAIR OF FUNCTIONS ARE INVERSES OF EACH OTHER.

A $f(x) = 2x - 4$; $g(x) = \frac{x+4}{2}$ **B** $f(x) = 2x + 5$; $g(x) = \frac{3x-5}{2}$